Comp. & Maths. with Appls. Vol. 12B, No. 5/6, pp. 1053–1059, 1986
Printed in Great Britain

0886-9561/86 \$3.00 + 0.00
Pergamon Journals Ltd

ERROR ESTIMATION FOR COLLOCATION SOLUTION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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(*Received* 29 *December* 1984)

Abstract—This paper is concerned with error estimates for the numerical solution of linear ordinary differential equations by global or piecewise polynomial collocation which are based on consideration of the differential operator involved and related matrices and on the residual. It is shown that a significant advantage may be obtained by considering the form of the residual rather than just its norm.

1. INTRODUCTION

This paper is concerned with computable error estimates for the solution of linear ordinary differential equations by global and piecewise polynomial collocation methods. This work is motivated by an abstract approach to error analysis such as described for example by Kantorovich and Akilov[1, Chap. XIV] and Anselone[2]. It extends previous work in this area by Cruickshank and Wright[3], with an emphasis here on error estimates while[3] concentrates on error bounds.

The underlying idea here is to make use of the matrix involved in the numerical solution in the error estimation process. The relationship between various matrices and the inverse of the differential operator has been considered by Wright[4], Gerrard and Wright[5] and Ahmed and Wright[6]. In particular these papers are concerned with asymptotic relationships between inverse operator norms and those of matrices related to the numerical solution. This theory leads naturally to the use of matrix norms as estimates for the corresponding operator norms.

Estimates for a residual norm lead immediately to estimates of the error in the solution. In the present paper, a number of modifications and improvements to this basic idea are considered, and it is shown that both less expensive and closer estimates are possible. This is done by considering not just the norm of the residual but also its form. The final estimate considered is similar though not identical to the idea of defect correction considered by Stetter[7]. Throughout this paper the analysis and illustrative results use infinity norms, though some of the ideas could be extended to other norms. The various algorithms are illustrated by using a selection of problems having different features. The examples do show the value of the algorithms and in particular the benefit of considering the form of the residual. The work presented here is based on the Ph.D. thesis of Ahmed[8].

2. ASSUMPTIONS AND NOTATION

In order to be able to treat both global and piecewise polynomial collocation in a uniform manner, slightly different notations from that of[6] will be used.

We consider the linear mth order differential equation of the form:

$$x^{(m)}(t) + \sum_{j=0}^{m-1} p_j(t)x^{(j)}(t) = y(t) \tag{1}$$

with m associated homogeneous boundary conditions. Without loss of generality we assume that the equation holds in $[-1, 1]$. The equation may be written in operator form:

$$(D^m - T)x = y \tag{2}$$

where D denotes the differentiation operator. In (2) we suppose that $x \in X$ and $y \in Y$ where X and Y are suitable Banach spaces. The operators $(D^m - T)$ and D^m with the associated conditions are both assumed to be invertible. The approximate collocation solution is taken in a subspace $X_{nq} \subset X$. To define this we first define a subspace $Y_{nq} \subset Y$. Suppose the interval $[-1, 1]$ is subdivided by the break-points $-1 = t_0 < t_1 < \dots t_n = 1$. In each subinterval q collocation points are used chosen as

$$\xi_{jk} = \{(t_k - t_{k-1})\,\xi_j^* + (t_k + t_{k-1})\}/2 \quad \begin{cases} j = 1, \dots, q \\ k = 1, \dots, n, \end{cases} \tag{3}$$

where $\{\xi_j^*\}$, $j = 1, \dots, q$, are given reference points in $[-1, 1]$. The space Y_{nq} consists of functions which are polynomials of degree $q - 1$ in each of the intervals $J_k = [t_{k-1}+, t_k-]$, $k = 1, \dots, n$. No assumptions regarding continuity at the break points is made. The solution space X_{nq} is then taken as $(D^m)^{-1}Y$. The projection operator ϕ_{nq} is defined as the operator which gives the interpolant in χ_{nq} based on the collocation points $\{\xi_{jk}\}$. With these assumptions the approximate solution x_{nq} satisfies

$$(D^m - \phi_{nq}T)x_{nq} = \phi_{nq}y. \tag{4}$$

In[6] certain matrices Q were introduced and their properties examined. Here we use a special case of this and denote it by Q_{nq}. This is most conveniently defined by considering the vector of values of the right hand side $\mathbf{y}$ and the solution $\mathbf{x}$ at the collocation points. Then there is a matrix Q_{nq} such that

$$\mathbf{x} = Q_{nq}\mathbf{y}$$

and this can be regarded as a definition of Q_{nq}. Under suitable conditions it was shown in[6] that

$$\|Q_{nq}\| \to \|(D^m - T)^{-1}\| \quad \text{as } n \to \infty, q \quad \text{fixed},$$

and

$$\|Q_{nq}\| \to \|(D^m - T)^{-1}\| \quad \text{as } q \to \infty,$$

where infinity norms we used in both cases. These conditions concerned the location of the collocation points and required the continuity of the coefficients $p_j(t)$ in (1). In particular the global case ($q \to \infty$, $n = 1$) assumed that the points $\{\xi_j^*\}$ were zeros of certain orthogonal polynomials. Full details of these assumptions are not included here as the results of[6] constitute only motivation for using the approximation

$$\|Q_{nq}\| \simeq \|(D^m - T)^{-1}\|. \tag{5}$$

The extra assumptions are not needed for the construction of the estimates considered below, though they might well be relevant to their quality. It is convenient here to define the compact operator K by

$$K = T(D^m)^{-1}. \tag{6}$$

In[4] and[5] matrices different from Q_{nq} were considered and these matrices were related directly to $(I - K)^{-1}$ rather than $(D^m - T)^{-1}$. These also provide (indirectly) estimates for $\|(D^m - T)^{-1}\|$, but since they are shown in[1] to be inferior to (5) we do not consider them further here.

3. THE RESIDUAL AND THE ERROR

Suppose an approximate solution x_{nq} of the differential equation (2) has been found and x_{nq} satisfies (4). Let the residual r_{nq} be defined by

$$r_{nq} = (D^m - T)x_{nq} - y \tag{7}$$

and the error by

$$e_{nq} = x_{nq} - x. \tag{8}$$

Using (2) we have

$$(D^m - T)\,e_{nq} = r_{nq} \quad \text{or} \quad e_{nq} = (D^m - T)^{-1}\,r_{nq}. \tag{9}$$

It immediately follows that

$$\|e_{nq}\| \leqslant \|(D^m - T)^{-1}\|.\|r_{nq}\|. \tag{10}$$

and in turn (5) suggests using the estimate

$$E_1 = \|Q_{nq}\|.\|r_{nq}\| \tag{11}$$

for the infinity norm of the error. Strictly this is an estimate of a bound on the error and so is likely to be larger than the error.

Note that r_{nq} may be evaluated at any point without difficulty since x_{nq} is a piecewise polynomial, and so $\|r_{nq}\|$ may be estimated by evaluation at a suitably fine grid of points.

The residual r_{nq} is constrained to be zero at the collocation points this implies that it will be of an oscillatory nature. Since also the operator $(D^m - T)^{-1}$ is essentially an integrating operator one would expect considerable cancellation in the evaluation (9) of e_{nq} which again suggests that the inequality (10) is likely to be crude. This in turn suggests taking into account the form of r_{nq}. A direct attempt to use the idea of defect correction with the approximation to the operator used in (4) is not useful as then only the values of the residual at the collocation points would be used and at these points the residual is zero. An alternative is to write the operator in a different form, as is often done in the treatment of integral equations.

Note first that

$$(D^m - T)^{-1} = (D^m)^{-1}\,(I - K)^{-1}, \tag{12}$$

this now allows the identity

$$(I - K)^{-1} = I + (I - K)^{-1}\,K \tag{13}$$

to be used giving

$$e_{nq} = (D^m)^{-1}\,\{I + (I - K)^{-1}\,K\}\,r_{nq} = (D^m)^{-1}r + (D^m - T)^{-1}\,Kr_{nq}. \tag{14}$$

Here $(D^m)^{-1}r_{nq}$ is an m fold integration where the given boundary conditions are satisfied. If we define

$$z_{nq} = (D^m)^{-1}r_{nq}$$

then

$$(Kr_{nq})(t) = (T\,z_{nq})(t) = -\sum_{j=0}^{m-1} p_j(t)z_{nq}^{(i)}\,t. \tag{15}$$

In general, it is necessary to make some further approximation to obtain a computable estimate. This may be done in a number of ways. One particularly convenient method is suggested by noting that if the coefficients $p_j(t)$ in the differential equation (1) are polynomial then will r_{nq} will be piecewise polynomial and in fact r_{nq} will have a factor

$$\prod_{j=1}^{q} (t - \xi_{jk})$$

in the kth subinterval. So if $p_j(t)$ are smooth r_{nq} should be well approximated by a piecewise polynomial found by interpolation using additional points in each subinterval. Clearly a considerable choice is available here and detailed suggestions are made later. If this interpolant, the "principal part of the residual" is denoted by r^*_{nq} we may write

$$r_{nq} = r^*_{nq} + r^{**}_{nq} \tag{16}$$

where r_{nq}^{**} is the error term in the interpolation. It is clearly straightforward to evaluate both $(D^m)^{-1}r_{nq}^*$ and Kr_{nq}^* as they involve only integration of piecewise polynomials, and then to estimate $\|(D^m)^{-1}r_{nq}\|$ and $\|Kr_{nq}\|$ by evaluation at a suitable selection of points. Using the estimate (5) for $\|(D^m-T)^{-1}\|$ then gives the following estimate for $\|e_{nq}\|$

$$E_2 = \|(D^m)^{-1}r_{nq}^*\| + \|Q_{nq}\|\{\|Kr_{nq}^*\| + \|r_{nq}^{**}\|\} \tag{17}$$

where $\|r_{nq}^{**}\|$ is also estimated by evaluation at a suitable choice of points. A simplified estimate can be obtained by ignoring $\|r_{nq}^{**}\|$ which should be valid if sufficient points are used to find r_{nq}^*. So we define

$$E_2^* = \|(D^m)^{-1}r_{nq}^*\| + \|Q_{nq}\| . \|Kr_{nq}^*\| . \tag{18}$$

A further alternative is to use an approximation to the operator $(D^m-T)^{-1}$ in (14) rather than its norm. This seems an appropriate generalization of the idea of defect correction and clearly it is now possible to use the original approximation $(D^m-\phi_{nq}T)^{-1}\phi_{nq}$. This makes the application particularly convenient, since the same matrix will be involved as in the original solution but with a new right-hand side, so that only a forward and back substitution are needed to solve the algebraic equations. This again gives rise to two estimates one including $\|r_{nq}\|$ one without this term.

Firstly define

$$e_{nq}^* = (D^m)^{-1}r_{nq}^* + (D^m-\phi_{nq}T)^{-1}\phi_{nq}Kr_{nq}^*,$$

then the estimates for $\|e_{nq}\|$ as

$$E_3 = \|e_{nq}^*\| + \|Q_{nq}\| . \|r_{nq}^{**}\| \tag{19}$$

and

$$E_3^* = \|e_{nq}^*\| . \tag{20}$$

This last estimate is particularly convenient and relatively cheap as the Q_{nq} matrix does not need to be constructed, since the construction of Q_{nq} requires nq extra forward and back substitutions using the decomposition of the original matrix, while finding e_{nq}^* requires only one. Note also that e_{nq}^* provides an estimate of the error as a function of t not just its norm. This implies that it could be used as a correction to the original solution, it also might be relevant for the construction of adaptive methods based on collocation. These points will, however, not be considered further in this paper.

4. PRACTICAL IMPLEMENTATION

To construct practical algorithms based on the estimates considered in Section 3 a number of specific choices need to be made both in the basic method and implementation.

Firstly, though any set of collocation points could be used we confine our attention (in the illustrative examples) to using either Chebyshev zeros or Gauss points as the reference points $\{\xi_j^*\}$ in (3). De Boor and Swartz[9] point out the improved convergence properties of Gauss points for piecewise collocation, and[4] and[6] suggest zeros of orthogonal polynomials for global collocation, and of these Chebyshev zeros are particularly convenient as they are easy to calculate.

Secondly, there is a need to decide the degree and choose the interpolation points for the principal part of the residual r_{nq}^*. One possibility is to find a relatively crude approximation, for example by using the end points of the subintervals as additional interpolation points (if they are not collocation points). A second possibility is to choose points between the collocation points, so as to get close to the extrema of the residual. With Chebyshev zeros this is again convenient as the extrema of $T_q(t)$ are at $\cos(j\pi/q), j=0,\ldots,q$. Using the orthogonality relationship satisfied by Chebyshev zeros at these points it is then convenient to express r_{nq}^* as

$$T_q(t^*)\sum_{r=0}^{q} a_r T_r(t^*), \tag{21}$$

where t^* denotes a local independent variable in each subinterval. An alternative which is convenient whatever the collocation points is to use the $2q+1$ points $\cos(j\pi/2q), j = 0, \ldots, 2q$, to give a representation in the form

$$\sum_{r=0}^{2q} b_r T_r(t^*). \tag{22}$$

For Chebyshev zero collocation points this is just an alternative representation of the same r^*_{nq}. Some idea of the accuracy of representation of r_{nq} can of course be obtained by examining the coefficients b_r.

This form is also convenient for carrying out the integrations needed to form $(D^m)^{-1} r^*_{nq}$ and its derivatives. These can be carried out first ignoring the arbitrary constant terms, which can then be obtained by setting up equations corresponding to the boundary and continuity conditions. This last choice has been used for the illustrative examples in the next section.

5. ILLUSTRATIVE EXAMPLES

Tables 1–5 display values for the estimates and some intermediate quantities for the following problems:

1. $x'' + 2x'/(t+3) + 2x/(t+3)^2 = -1/(t+3), x(\pm 1) = 0,$
2. $x'''' + x''' + \exp(t)x = 100, x(\pm 1) = x'(\pm 1) = 0,$
3. $x'' - x = 1/(t^2 + 0.1), x(\pm 1) = 0,$
4. $x'' + |t|x = 1, x(\pm 1) = 0,$
5. $x'' - 100(2 - t^2)x = 100, x(\pm 1) = 0.$

The number of subintervals is indicated by n and the number of collocation points in each subinterval by q, the letters T and G are used to indicate whether (shifted) Chebyshev zeros or Gauss points have been used. The problems are all scaled so that the maximum value of the solution is roughly of order 1. The first two problems have smooth coefficients, and are of order two and four respectively. Problem 3 has smooth coefficients but the right-hand side and solution are rapidly varying near $t = 0$. In problem 4 the $p_j(t)$ coefficient has a discontinuous derivative at $t = 0$. Problem 5 has a large coefficient $p_j(t)$, this results in the solution having mild boundary layers near the two end points. This problem also has polynomial coefficients and right-hand side so that $r^{**} = 0$, $E_2 = E_2^*$ and $E_3 = E_3^*$.

Table 1. Problem 1: $x'' + 2x'/(t+3) + 2x/(t+3)^2 = -1/(t+3)$

n	q	$\lVert r \rVert$	$\lVert r^{**} \rVert$	$\lVert (D^m)^{-1} r^* \rVert$	$\lVert Kr^* \rVert$	E_1	E_2	E_2^*	E_3	E_3^*	$\lVert e \rVert$
1	3T	3.48′−2	3.53′−5	3.31′−3	1.06′−2	1.92′−2	9.19′−3	9.18′−3	3.00′−3	2.98′−3	2.84′−3
1	6T	1.16′−4	4.71′−9	3.27′−6	1.07′−5	6.38′−5	9.17′−6	9.17′−6	3.41′−6	3.41′−6	3.38′−6
1	9T	1.16′−6	2.46′−13	8.16′−9	5.58′−8	6.44′−7	3.90′−8	3.90′−8	8.17′−9	8.17′−9	8.10′−9
1	12T	2.04′−8	6.29′−16	9.22′−11	7.55′−10	1.15′−8	5.16′−10	5.16′−10	9.36′−11	9.36′−11	9.28′−11
3	3T	2.70′−3	1.94′−7	6.35′−5	2.13′−4	1.49′−3	1.82′−4	1.82′−4	7.58′−5	7.57′−5	7.40′−5
6	3T	4.16′−4	3.18′−9	3.61′−6	1.81′−3	2.33′−4	1.38′−5	1.38′−5	4.12′−6	4.11′−6	4.03′−6
1	3G	5.37′−2	3.46′−5	1.93′−3	3.54′−3	2.96′−2	3.90′−3	3.88′−3	1.76′−3	1.74′−3	1.67′−3
3	3G	4.30′−3	1.94′−7	1.72′−5	1.07′−4	2.38′−3	7.66′−5	7.65′−5	1.80′−5	1.79′−5	1.70′−5
6	3G	6.65′−4	3.18′−9	6.61′−7	8.48′−6	3.72′−4	5.42′−6	5.42′−6	7.16′−7	7.15′−7	6.67′−7

Table 2. Problem 2: $x'''' + x''' + \exp(t)x = 100$

n	q	$\lVert r \rVert$	$\lVert r^{**} \rVert$	$\lVert (D^m)^{-1} r^* \rVert$	$\lVert Kr^* \rVert$	E_1	E_2	E_2^*	E_3	E_3^*	$\lVert e \rVert$
1	3T	4.75′ 0	1.24′−2	2.02′−2	1.68′ 0	1.85′−1	8.60′−2	8.55′−2	2.24′−2	2.20′−2	2.41′−2
1	6T	3.70′−2	2.14′−8	4.34′−5	3.24′−3	1.33′−3	1.59′−4	1.59′−4	3.66′−6	3.66′−6	3.65′−5
1	9T	4.20′−5	9.05′−14	7.42′−9	2.54′−6	1.64′−6	1.07′−7	1.07′−7	7.22′−9	7.22′−9	7.21′−9
1	12T	1.59′−7	1.35′−13	6.22′−12	6.18′−9	6.15′−9	2.46′−10	2.46′−10	6.30′−12	6.30′−12	6.42′−12
3	3T	2.83′−1	1.64′−5	1.93′−4	3.00′−2	1.12′−2	1.38′−3	1.38′−3	1.72′−3	1.71′−3	1.58′−4
6	3T	4.87′−2	1.54′−7	8.81′−6	1.85′−3	1.91′−3	8.16′−5	8.16′−5	7.57′−6	7.56′−6	6.76′−6
1	3G	7.50′ 0	1.27′−2	1.24′−2	5.74′−1	2.92′−1	3.53′−2	3.48′−2	1.26′−2	1.21′−2	1.48′−2
3	3G	4.27′−1	1.64′−5	4.55′−5	1.21′−2	1.67′−2	4.84′−4	4.83′−4	4.49′−5	4.43′−5	5.01′−7
6	3G	5.61′−2	1.55′−7	9.43′−7	8.55′−4	2.20′−3	3.45′−5	3.45′−5	9.58′−7	9.52′−7	1.16′−6

Table 3. Problem 3: $x'' - x = 1/(t^2 + 0.1)$

n	q	$\|r\|$	$\|r^{**}\|$	$\|(D^m)^{-1}r^*\|$	$\|Kr^*\|$	E_1	E_2	E_2^*	E_3	E_3^*	$\|e\|$
1	3T	4.24′ 0	1.43′ 0	9.64′−1	1.00′ 0	1.49′ 0	1.82′ 0	1.32′ 0	1.18′ 0	6.78′−1	8.36′−1
1	6T	3.01′ 0	2.07′−1	2.83′−1	2.68′−1	9.94′−1	4.40′−1	3.71′−1	2.87′−1	2.19′−1	2.12′−1
1	9T	5.51′−1	3.34′−2	4.17′−2	4.37′−2	1.94′−1	6.88′−2	5.71′−2	4.34′−2	3.17′−2	3.19′−2
1	12T	4.77′−1	5.15′−5	9.67′−3	8.90′−3	1.65′−1	1.45′−2	1.27′−2	9.89′−3	8.10′−3	8.09′−3
3	3T	7.15′−1	3.11′−2	8.47′−2	8.53′−2	2.52′−1	1.26′−1	1.15′−1	7.26′−2	6.16′−2	6.19′−2
6	3T	2.21′−1	4.60′−4	5.51′−3	5.21′−3	7.77′−2	7.15′−3	6.99′−3	4.29′−3	4.13′−3	4.12′−3
1	3G	4.82′ 0	1.43′ 0	7.59′−1	7.95′−1	1.70′ 0	1.54′ 0	1.04′ 0	1.04′ 0	5.32′−1	6.97′−1
3	3G	1.30′ 0	3.11′−2	2.83′−2	2.90′−2	4.57′−1	4.95′−2	3.85′−2	3.24′−2	2.15′−2	2.18′−2
6	3G	3.62′−1	4.60′−4	9.12′−4	8.68′−4	1.27′−1	1.38′−3	1.22′−3	9.27′−4	7.64′−4	7.64′−4

Table 4. Problem 4: $x'' + |t|x = 1$

n	q	$\|r\|$	$\|r^{**}\|$	$\|(D^m)^{-1}r^*\|$	$\|Kr^*\|$	E_1	E_2	E_2^*	E_3	E_3^*	$\|e\|$
1	3T	1.56′−1	5.21′−2	3.65′−2	1.61′−2	7.99′−2	7.14′−2	4.47′−2	6.43′−2	3.76′−2	5.42′−2
1	6T	1.08′−1	2.87′−2	1.12′−2	2.60′−3	5.82′−2	2.80′−2	1.26′−2	2.79′−2	1.24′−2	8.68′−3
1	9T	3.82′−2	1.86′−2	4.08′−3	1.07′−3	2.14′−2	1.51′−2	4.68′−2	1.49′−2	4.45′−3	6.03′−3
1	12T	4.86′−2	1.41′−2	2.53′−3	6.30′−4	2.72′−2	1.08′−2	2.88′−3	1.07′−2	2.78′−3	1.89′−3
3	3T	4.20′−2	1.82′−2	4.81′−3	1.37′−3	2.35′−2	1.58′−2	5.58′−3	1.55′−2	5.27′−3	6.87′−3
6	3T	7.14′−4	4.37′−16	2.12′−5	5.76′−6	4.04′−4	2.45′−5	2.45′−5	2.31′−5	2.31′−5	2.31′−5
1	3G	2.77′−1	5.31′−2	2.72′−2	1.12′−2	1.45′−1	6.08′−2	3.30′−2	5.63′−2	2.85′−2	4.37′−2
3	3G	5.58′−2	1.83′−2	2.80′−3	7.25′−4	3.14′−2	1.35′−2	3.21′−3	1.34′−2	3.06′−3	4.62′−3
6	3G	1.01′−3	4.11′−16	1.16′−6	6.78′−7	5.72′−4	1.55′−6	1.55′−6	1.17′−6	1.17′−6	1.17′−6

Table 5. Problem 5: $x'' - 100\,(2 - t^2) = -100$

n	q	$\|r\|$	$\|(D^m)^{-1}r^*\|$	$\|Kr^*\|$	E_1	E_2	E_3	$\|e\|$
1	3T	7.18′ 1	1.44′ 1	2.93′ 3	4.98′−1	3.50′ 1	1.15′ 0	3.49′−1
1	6T	3.07′ 1	1.14′ 0	1.99′ 2	2.07′−1	2.48′ 0	4.80′−1	1.30′−1
1	9T	2.66′−1	2.65′−3	4.41′−1	1.75′−3	5.56′−3	8.23′−4	5.72′−4
1	12T	1.09′−1	7.99′−4	1.30′−1	7.38′−4	1.67′−3	6.01′−4	3.18′−4
3	3T	2.37′ 1	5.66′−1	1.09′ 2	1.54′−1	1.28′ 0	1.69′−1	1.02′−1
6	3T	4.81′ 0	4.05′−2	8.08′ 0	3.22′−2	9.46′−2	1.14′−2	9.72′−3
1	3G	8.09′ 1	6.00′ 0	1.27′ 3	5.38′−1	1.44′ 1	4.67′−1	1.53′−1
3	3G	3.12′ 1	1.37′−1	2.56′ 1	2.00′−1	3.01′−1	8.41′−2	5.75′−2
6	3G	7.25′ 0	7.74′−3	1.09′ 0	4.68′−2	1.48′−2	6.64′−3	5.93′−3

The norm values given in the tables were estimated by evaluation at 200 equispaced points. The norm of the error was estimated by comparison with a more accurate solution.

In problems 1 and 2 $\|r^{**}\|$ is small which indicates that r^* is a good approximation for the residual. This approximation improves as the number of points increases for both global and piecewise approximation. The error estimates E_3 and E_3^* are also all reasonably close again with improvement as the number of points increases. For problem 2, E_1 and E_2 significantly overestimate the error in some cases, this is not surprising as they are estimates of bounds.

In problem 3 $\|r^{**}\|$ is not so small though again this decreases relative to r as the number of points increases. Even though the solution is quite poor for small number of points the estimates are reasonably satisfactory. E_1 and E_2 again overestimate the error significantly in some case. E_3^* on the other hand is very close still even though it occasionally underestimates the error slightly. This is again not surprising as $\|r^{**}\|$ is ignored here.

For problem 4 $\|r^{**}\|$ is relatively large except when $n = 6$. In this last case a break point occurs at the point of discontinuity of the derivative of $|t|$, so that in each subrange the coefficients are polynomial and $r^{**} = 0$. The error in the $n = 6$ case is significantly smaller than for other choices, this accords with the result given by Russell and Shampine[10] who point out the advantage of having break points or points were the coefficients have discontinuities in derivatives. The differences in accuracy are clearly reflected in the error estimates. Otherwise the results are generally similar to those for problem 3. Problem 5 was chosen so that K is large and it is clear that $\|Kr^*\|$ is greater than $\|r\|$. It follows that the second term in the expression for E_3^* (1) will be dominant. In all other cases $\|(D^m)^{-1}r^*\|$ is itself a reasonable approximation for the error. It is interesting to note that for this problem E_1 is the best estimate of the error when the number of collocation points is small. It should also be noted that the error estimates are still reasonably reliable even when the solution is very poor.

Comparing the results for piecewise collocation using Gauss and Chebyshev points the higher accuracy of the solution using Gauss points pointed out by De Boor and Swartz[9] is observed. This is also reflected in the error estimates, except occasionally for the E_1 estimates.

6. CONCLUSIONS

The error estimation techniques described in this paper have been shown to be all effective at least for the limited selection of examples given. The estimates E_3 and E_3^* in particular seem particularly good, being both fairly inexpensive and giving close bounds.

Clearly a more extensive comparison both on a wider selection of problems and with alternative estimates would be valuable. There is some difficulty, however, in making an assessment, as many minor variants of the methods are possible and these could affect both the amount of work involved and the reliability of the estimates.

Alternative algorithms which could be considered include estimates based on consistency of independent solutions involving different numbers of collocation points and estimates based on the size of Chebyshev series coefficients. Estimates of this type, such as considered by Delves[11] for example, are particularly cheap if the solution is represented in such a form. For piecewise collocation algorithms using information from different subintervals are available as given for example by de Boor[12] and used by Russell and Christiansen[13]. In these last papers the emphasis is on mesh selection rather than just error estimation, but the estimates given there could be used for this purpose, though values given there as generic constants would need to be given specific values. On the other hand the function on which E_3 is based could be used in mesh selection algorithms. It is hoped to consider this possibility in the future.

REFERENCES

1. L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Mormed Spaces*. Pergamon Press, New York (1964).
2. P. M. Anselone, *Collectively Compact Operator Approximation Theory*. Prentice-Hall, Englewood Cliffs, N.J. (1971).
3. D. M. Cruickshank and K. Wright, Computable error bounds for polynomial collocation methods. *SIAM Jl Num. Anal.* **15**, 134–151 (1978).
4. K. Wright, Asymptotic properties of collocation matrix norms 1: global polynomial approximation. *I.M.A. Jl Num. Anal.* **4**, 185–202 (1984).
5. C. Gerrard and K. Wright, Asymptotic properties of collocation matrix norms 2: piecewise polynomial approximation. *I.M.A. Jl Num. Anal.* **4**, 363–373 (1984).
6. A. H. Ahmed and K. Wright, Further asymptotic properties of collocation matrix norms. *I.M.A. Jl Num. Anal.* **5**, 235–246 (1985).
7. H. J. Stetter, The defect correction principle and discretization methods. *Num. Math.* **29**, 425–443 (1978).
8. A. H. Ahmed, Collocation algorithms and error analysis for approximate solutions of ordinary differential equations. Ph.D. thesis, University of Newcastle upon Tyne (1981).
9. C. de Boor and B. Swartz, Collocation at Gaussian points. *SIAM Jl Num. Anal.* **10**, 582–606 (1973).
10. R. D. Russell and L. F. Shampine, A collocation method for boundary value problems. *Num. Math.* **19**, 1–28 (1972).
11. L. M. Delves, in *Modern Numerical Methods for Ordinary Differential Equations* (Edited by G. Hall and J. M. Watt), Chap. 19. Oxford University Press (1976).
12. C. de Boor, Good approximation by splines with variable knots II. In *Proceedings of a Conference on the Numerical Solution of Differential Equations* (Edited by G. A. Watson), *Lecture Notes in Mathematics 363.* Springer, Berlin (1974).
13. R. D. Russell and J. Christiansen, Adaptive mesh selection strategies for solving boundary value problems. *SIAM Jl Num. Anal.* **15**, 59–80 (1978).