

ASYMPTOTIC PROPERTIES OF COLLOCATION PROJECTION NORMS

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Abstract—This paper is concerned with the behaviour of certain collocation projection norms. It is shown that the norms of the piecewise collocation projection and the usual piecewise interpolating projection are asymptotically the same. A similar result is shown for the global case if the collocation points are chosen as Tchebycheff zeros with some restrictions on the coefficients of the differential operator.

1. INTRODUCTION

This paper extends the analysis of collocation methods for the solution of ordinary differential equations using operator methods described by Cruickshank and Wright [1], Wright [2], Gerard and Wright [3] and Ahmed and Wright [4], where references to related work (e.g. Karpilovskaja [5, 6], Vanikko [7], Kantorovick and Akilov [8, Chap. XIV] and Anselone [9]) may be found. Here it is shown that the norm of the projection defined with the solution of the differential equation using piecewise collocation method is asymptotically the same as the norm of the piecewise interpolating projection. A similar result holds for the global case if the collocation points are chosen as Tchebycheff zeros and if the coefficients of the differential equation are restricted to be polynomials of certain degrees. These results are useful in error analysis of collocation methods; in particular they improve the applicability of the error bounds described in Ref. [1] and thesis [10].

Before investigating the results more precisely we introduce the following assumptions and notations.

We consider an m th order differential equation of the form

$$x^{(m)}(t) + \sum_{j=0}^{m-1} P_j(t)x^{(j)}(t) = y(t), \quad (1)$$

with m associated homogeneous boundary conditions. This may be written in the operator form:

$$(D^m - T)x = y, \quad (2)$$

where $(D^m x)(t) = (d^m x/dt^m)(t)$. In equation (2) we suppose that x is in X and y is in Y , where X and Y are suitable Banach spaces. The operator $D^m - T$ denoted by D^* , with the associated conditions is assumed to be invertible.

Let X_n and Y_n be subspaces of X and Y , respectively and ϕ_n a projection: $Y \rightarrow Y_n$. An approximate solution $x_n \in X_n$ is found (where possible) by applying the projection ϕ_n to equation (1) with x_n substituted for x , that is

$$\phi_n(D^m x_n - T x_n - y) = 0. \quad (3)$$

The subscript n will be related to the dimension of the subspaces X_n and Y_n which are assumed equal. It is also assumed that $\phi_n D^m x_n = D^m x_n$, that is, D^m restricted to X_n establishes a bijection between X_n and Y_n . It follows that x_n satisfies

$$(D^m - \phi_n T)x_n = \phi_n y. \quad (4)$$

It is assumed that $K = T(D^m)^{-1}$ is a compact operator. Without loss of generality we assumed that the equation is satisfied in the range $[-1, 1]$. The coefficients $P_j(t)$ in equation (1) are assumed to be continuous in the range. The space Y will be taken as $R[-1, 1]$ the space of Riemann integrable functions on $[-1, 1]$, as approximate solutions with finite discontinuities in their m th

derivatives are considered. X is taken as $D^{-1}Y$, that is, the set of functions satisfying the boundary conditions for which $D^m X = Y$. The space Y_n will be taken to be a polynomial or piecewise polynomial subspace defined below, and the infinity norm is used throughout.

For global polynomial collocation Y_n is taken as the space of polynomials of degree $n - 1$ on $[-1, 1]$ and ϕ_n is the polynomial projection based on collocation points $\{v_j\}, j = 1, \dots, n$. For piecewise polynomial collocation the range is divided into n subintervals by the breakpoints $-1 = t_0 < t_1 \cdots t_n = 1$. In each subinterval p collocation points are used, chosen as

$$v_{jk} = ((t_k - t_{k-1})v_j^* + (t_k + t_{k-1}))/2 \quad j = 1, \dots, p, k = 1, \dots, n,$$

where $\{v_j^*\}, j = 1, \dots, p$ are given reference points in $[-1, 1]$. The space Y_n consists of functions which are polynomials of degree $p - 1$ in each of the intervals $J_i = [t_{i-1}, t_i], i = 1, \dots, n$. The projection ϕ_n is the corresponding interpolation projection based on these collocation points, which is equivalent to polynomial interpolation in each subinterval. Both the set of points $\{v_j\}$ and $\{v_j^*\}$ are assumed to be in ascending order.

As in Refs [1, 2, 4], the global collocation points are assumed to be zeros of some orthogonal polynomial $Q_n(t)$ with weight function $P(t)$. For piecewise polynomial collocation it is assumed that the points $\{v_j^*\}$ are chosen so that the corresponding interpolatory quadrature weights are positive as in Ref. [3].

We introduce Y_n^* to be the space of polynomials or piecewise polynomials generated by $\{D^*z_r\}$, where $\{z_r\}, r = 1, \dots, n$, is a basis of X_n . Now if D^{*-1} is well defined then for the piecewise case, D^* establishes a bijection between X_n and $Y_n^* = \phi_n^* Y$. Thus, $\phi_n^* D^* x_n = D^* x_n$, for all x_n in X_n and ϕ_n^* defines a linear projection from Y to Y_n^* . The same bijection can be established for the global case, but only if the coefficients of the differential equation are restricted in such a way that the j th coefficient $P_{j(t)}$ is a polynomial of degree j .

It follows then, that x_n , satisfies,

$$D^* x_n = \phi_n^* y, \quad (5)$$

which will imply from equation (4) that

$$D^{*-1} \phi_n^* y = (D^m - \phi_n T)^{-1} \phi_n y.$$

If we multiply the left-hand side by D^* we get

$$\phi_n^* y = D^*(D^m - \phi_n T)^{-1} y.$$

Hence

$$\begin{aligned} \phi_n^* &= D^*(D^m - \phi_n T)^{-1} \phi_n, \\ &= (D^m - T)(D^m - \phi_n T)^{-1} \phi_n \\ &= (I - TD^{m-1})D^m D^{m-1}(I - \phi_n TD^{m-1})\phi_n \\ &= (I - K)(I - \phi_n K)^{-1} \phi_n. \end{aligned} \quad (6) \quad (7)$$

The result to be considered in the next section is that

$$\|\phi_n^* - \phi_n\| \rightarrow 0,$$

provided

$$\max_i |t_i - t_{i-1}| \xrightarrow{i=1,2,\dots,n} 0$$

in the piecewise case and the collocation points are chosen as Tchebycheff zeros for the global case.

We state now the following identity which may be required several times in the proof of the result

$$\phi_n(I - K\phi_n)^{-1} = (I - \phi_n K)^{-1} \phi_n, \quad (8)$$

for

$$\begin{aligned} (I - \phi_n K)\phi_n &= (I + \phi_n K(I - \phi_n K)^{-1})\phi_n \\ &= \phi_n(I + K(I - \phi_n K)^{-1})\phi_n. \end{aligned}$$

but from Ref. [1]

$$(I - K\phi_n)^{-1} = I + K(I - \phi_n K)y,$$

Therefore

$$(I - \phi_n K)\phi_n = \phi_n(I - K\phi_n)^{-1}.$$

2. RELATION BETWEEN THE TWO PROJECTION NORMS

From equation (7)

$$\begin{aligned}\phi_n^* - \phi_n &= (I - K)(I - \phi_n K)^{-1}\phi_n - \phi_n \\ &= (I - K)\phi_n(I - K\phi_n)^{-1} - \phi_n\end{aligned}$$

by equation (8)

$$\begin{aligned}&= \phi_n(I - K\phi_n)^{-1} - K\phi_n(I - K\phi_n)^{-1} - \phi_n \\ &= \phi_n(I - K\phi_n)^{-1} - (I - K\phi_n)^{-1} + I - \phi_n \\ &= (I - \phi_n)(I - (I - K\phi_n)^{-1}) \\ &= (\phi_n - I)K\phi_n(I - K\phi_n)^{-1} \\ &= (\phi_n - I)K\phi_n(I - K\phi_n)^{-1}\end{aligned}$$

or

$$\|\phi_n^* - \phi_n\| \leq \|(\phi_n - I)K\phi_n\| \|(I - K\phi_n)^{-1}\|.$$

Therefore

$$\|\phi_n^* - \phi_n\| \rightarrow 0, \quad (9)$$

if we show that $\|(\phi_n - I)K\phi_n\| \rightarrow 0$ since by Ref. [1], $\|(I - K\phi_n)^{-1}\|$ is uniformly bounded.

For the piecewise case,

$$\|(\phi_n - I)K\phi_n\| \leq \|(\phi_n - I)K\| \|\phi_n\| \rightarrow 0,$$

since $\|\phi_n\|$ is a constant and $\|(\phi_n - I)K\| \rightarrow 0$, as shown in Ref. [3].

For the global case, it is shown in Ref. [1] using Jackson's theorem [11] that

$$\|(I - \phi_n)K^d\| \leq \frac{\|I - \phi_n\| \|D^d K\|}{n(n-1)\cdots(n-d+1)};$$

and therefore for $d = 1$

$$\|(I - \phi_n)K\| \leq \frac{\|I - \phi_n\| \|DK\|}{n}$$

or

$$\begin{aligned}\|(I - \phi_n)K\phi_n\| &\leq \frac{\|I - \phi_n\| \|DK\| \|\phi_n\|}{n} \\ &\leq \frac{(\|\phi_n\| + 1)\|DK\| \|\phi_n\|}{n} \rightarrow 0.\end{aligned}$$

If $\|\phi_n\|^2/n \rightarrow 0$; which is guaranteed if the collocation points are chosen as Tchebycheff points, where $\|\phi_n\| \sim 0(Ln)$ [12, p. 52].

We show next that for small $n\|\phi_n^*\|$ is not going to be very much worse than $\|\phi_n\|$. For, from equation (8)

$$\phi_n^* = (I - K)\phi_n(I - K\phi_n)^{-1}.$$

Therefore

$$\|\phi_n^*\| \leq (\|\phi_n\| + \|K\phi_n\|)\|(I - K\phi_n)^{-1}\|.$$

However, from Ref. [1],

$$\|(I - K\phi_n)^{-1}\| \leq \|W_n\| \|K\phi_n\| + 1,$$

where $\|W_n\|$ is the matrix mapping the values of y at the collocation points to those of $x^{(m)}$ at these same points. Therefore,

$$\|\phi_n^*\| \leq (\|\phi_n\| + \|K\phi_n\|)\|W_n\| \|K\phi_n\|. \quad (10)$$

$\|K\phi_n\|$ can be bounded by $\|K\| \|\phi_n\|$ (or $\pi^2 \max_{s,t} k(s,t)$ for the global case [1]).

3. CALCULATION OF THE PROJECTION NORM $\|\phi_n^*\|$

(a) *The global case*

To calculate $\|\phi_n^*\|$ for the global case we express $(\phi_n^* y)(t)$ as

$$\sum_{j=1}^n L_j(t)y(v_j),$$

where $L_n(t)$ are the generalized Lagrange interpolating coefficients. Then,

$$\|\phi_n^*\| = \sup_t \sum_{j=1}^n |L_j(t)|$$

as usual.

By definition, $L_j(v_k) = \delta_{jk}$ and may be expressed as

$$L_j(t) = \phi_n^* z e_j,$$

where e_j is the j th unit vector and z is an extension operator from R^n to Y giving a function on Y whose values at the collocation points agree with the components of the vector.

Now returning to equation (5) we see that $L_j(t)$ can be found by solving

$$D^* x^{(j)} = z e_j, \quad (11)$$

by collocation points $\{v_j\} j = 1, \dots, n$. If its numerical solution is $x^{(j)}(t)$, then clearly

$$L_j(t) = D^* x^{(j)}(t).$$

This may well be the simplest way of calculating $L_j(t)$ and hence $\|\phi_n^*\|$.

(b) *The piecewise case*

In this case the projection ϕ_n^* is defined by

$$(\phi_n^* y)(t) = \sum_{j=1}^{np} L_j(t)y(v_j),$$

where L_j is the unique polynomial such that $L_i(v_j) = \delta_{ij}$ and $[v_j]_{j=1, \dots, np}$ are given by $v = v_{jk}$, $k = 1, \dots, n$, $(k-1)p + j$, $j = 1, \dots, p$.

We note here that ϕ_n^* cannot be defined in each interval (t_{i-1}, t_i) separately as in the case of ϕ_n where continuity of the m th derivative is not required.

To calculate $L_j(t)$, as in the global case we solve

$$D^* x^{(j)} = z e_j \quad j = 1, \dots, np,$$

by piecewise collocation method using the points v_{jk} , $i = 1, \dots, n$, $j = 1, \dots, p$. Then

$$L_j(t) = D^* x^{(j)}(t)$$

and

$$\|\phi_n^*\| = \max_t \sum_{j=1}^{np} |L_j(t)|.$$

Table 1. ϕ^* Global (negative c)

$n/c =$	-1×10^{-5}	-0.5	-10	-50	0
5	1.98885	1.97824	1.82248	1.88896	1.98885
10	2.42883	2.42593	2.37461	2.21149	2.42883
15	2.68671	2.68539	2.66094	2.57132	2.68671
20	2.86977	2.86902	2.85486	2.79987	2.86977
25	3.01179	3.01130	3.00210	2.86532	3.01179
30	3.12784	3.12750	3.12104	3.0984	3.12784

Table 2. ϕ^* Global (positive c)

$n/c =$	1×10^{-5}	0.5	9.869604	50	0
5	1.98885	1.9998	5.42223	1.99848	1.98885
10	2.42883	2.43175	1.44×10^3	2.92604	2.42883
15	2.68671	2.68805	2.71379	2.64581	2.68671
20	2.86977	2.87053	2.88501	2.95305	2.86977
25	3.01179	3.07228	3.02157	3.06370	3.01179
30	3.12784	3.12818	3.13465	3.16345	3.12784

4. ILLUSTRATION

For illustration we define D^* by

$$D^*x = x'' + cx, \quad x(\pm 1) = 0.$$

$\|\phi_n^*\|$ is calculated for different values of c for both the global and piecewise cases. The results for the global case for negative and positive values of c are presented in Tables 1 and 2, respectively. We observe that for all values of c when n is sufficiently large $\|\phi_n^*\|$ takes similar values each of which is the limit of $\|\phi_n\|$ when $n \rightarrow 0$ for Tchebycheff zeros. We also observe that the effect of singularity (for example $c = 869604$) is less than one would expect. This may be due to the approximations involved and the cancellations occurring in multiplying back by D^* in equation (11).

Results for the piecewise case for negative and positive values of c are presented in Tables 3 and 4, respectively. We observe that except for $c = 9.869604$ where singularity is expected, $\|\phi_n^*\|$ is always tending to the same value. This here is 1.414213 which is the value of $\|\phi_n\|$ when the collocation points are two Tchebycheff zeros.

5. CONCLUSION

The results considered in this paper extend the usual interpolating projection operator and make the choice of the projection in the collocation methods more flexible. One of the practical uses of the results, for example, is to put D^* in equation (2) in the form

Table 3. ϕ_n Piecewise, two points (negative c)

$n/c =$	-1×10^{-5}	-0.5	-10	-50	-500
5	1.414213	1.412528	1.421329	1.423347	1.858299
10	1.414213	1.413833	1.419330	1.430870	1.555555
15	1.414213	1.414051	1.417218	1.426088	1.421426
20	1.414213	1.414124	1.416134	1.422605	1.430752
25	1.414213	1.414157	1.415541	1.420364	1.432364
30	1.414213	1.414174	1.415181	1.418887	1.431551

Table 4. ϕ^* Piecewise, two points (positive c)

$n/c =$	1×10^{-5}	0.5	9.869602	50	500
5	1.414213	1.417604	4.800538	3.869048	3.168807
10	1.414213	1.415125	4.798982	2.325582	65.11853
15	1.414213	1.414629	4.833981	1.859074	15.40038
20	1.414213	1.414450	4.920161	1.672735	8.694045
25	1.414213	1.414366	4.991944	1.579949	10.13541
30	1.414213	1.414320	5.029934	1.530478	19.2358

$$(D^*x)(t) = (d^m x)/(dt^m)(t) - \sum_{k=m-1}^0 c_k (d^k x)/(dt^k)(t),$$

where $\{c_k\}_{k=0, \dots, m-1}$ are chosen to give the highest possible applicability of the error bounds derived in Cruickshank and Wright [1] and Ahmed [10].

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